Introduction to Mathematical Quantum Theory

Text of the Exercises

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Exercise 1

Let $\mathcal{H} = L^2(\mathbb{R}^2)$. Let $\widetilde{H}$ be defined as

$$\widetilde{H} := -\frac{1}{2}(\Delta_x + \Delta_y) + \frac{1}{2}(x^2 + y^2) - \lambda xy \tag{1}$$

with $D\left(\widetilde{H}\right) = C_c^\infty(\mathbb{R}^2)$.

Prove that if $\lambda \in (-1, 1)$ then $\widetilde{H}$ is essentially self adjoint and study the spectrum of the closure of $\widetilde{H}$.

Hint: Prove that, with the right change of variables $(x, y) \to (w, z)$, $\widetilde{H} = H_w + H_z$ with H_w only depending on w and H_z only depending on z.

Exercise 2

Let A be a normal matrix (meaning that $AA^* = A^*A$) and p a polynomial in two variables. Show by example that an eigenvector for $p(A, A^*)$ is not necessarily an eigenvector for A.

Remark: Even if eigenvectors of $p(A, A^*)$ do not correspond to eigenvectors of A, the spectrum does, in the sense that

$$\sigma(p(A, A^*)) = \{p(\lambda, \lambda^*) \mid \lambda \in \sigma(A)\}. \tag{2}$$

Exercise 3

Let $I := [0, 1]$ and consider $\mathcal{H} = L^2(I)$. Define the operator $H := -\Delta$ with domain[1] $D(H) := H^2(I) \cap C^1_{\text{per}}(I)$. Prove that H is self-adjoint and exhibit its spectral measure explicitly.

Exercise 4

Let $\mathcal{H}$ be an Hilbert space and $A_+,\ A_- \in \mathcal{B}(\mathcal{H})$ such that

$$\left[A_\pm, A_\pm^*\right] = \text{id}, \tag{3}$$

$$\left[A_+, A_-\right] = \left[A_+, A_-^*\right] = 0. \tag{4}$$

[1]This definition makes sense, because we know that for any function $\psi \in H^2(I)$ we have that there is a function $\widetilde{\psi} \in C^1(I)$ that coincides almost everywhere with ψ. The definition of the domain is then the set of functions $\psi \in H^2(I)$ such that the function $\widetilde{\psi}$ is periodic with derivative which is periodic.

Let moreover $\eta,\ \zeta \in \mathbb{R}$, with $\eta > \zeta \geqslant 0$. Define

$$H := \eta\left(A_+^* A_+ + A_-^* A_-\right) + \zeta\left(A_+^* A_-^* + A_+ A_-\right). \tag{5}$$

Recall that if $\theta = \frac{1}{2}\operatorname{arctanh}\left(\frac{\zeta}{\eta}\right)$, $\alpha = \sqrt{\eta^2 - \zeta^2}$, $\beta = \sqrt{\eta^2 - \zeta^2} - \eta$ and C_+ and C_- are defined as

$$C_\pm := \cosh(\theta) A_\pm + \sinh(\theta) A_\mp^* \tag{6}$$

we get

$$\left[C_\pm, C_\pm^*\right] = \mathrm{id}, \tag{7}$$

$$\left[C_+, C_-\right] = \left[C_+, C_-^*\right] = 0, \tag{8}$$

$$H = \alpha\left(C_+^* C_+ + C_-^* C_-\right) + \beta. \tag{9}$$

a Consider $X := A_+^* A_-^* - A_+ A_-$. Prove that X is skew-adjoint, meaning that $X^* = -X$.

b For any $t \in \mathbb{R}$ consider $U(t) := e^{-tX}$. Prove that $\{U(t)\}_{t\in\mathbb{R}}$ is a strongly continuous one-parameter unitary group such that

$$U(t) A_\pm U(-t) = \cosh(t) A_\pm + \sinh(t) A_\mp^*. \tag{10}$$

Hint: Consider for any $\psi, \varphi \in \mathcal{H}$ the function $f : \mathbb{R} \to \mathbb{R}$ defined as

$$f_\pm(t) := \langle \psi, U(t) A_\pm U(-t) \varphi \rangle. \tag{11}$$

Prove that f satisfies a closed second order differential equation and deduce (10).

c Suppose that there is a complete orthonormal system $\{\varphi_n\}_{n\in\mathbb{N}}$ for $\mathcal{H}$ such that $A_\pm^* A_\pm \varphi_n = \epsilon_n^\pm \varphi_n$, with $\epsilon_n^\pm \in \mathbb{R}$. Prove that there exist a complete orthonormal system $\{\psi_n\}_{n\in\mathbb{N}}$ for $\mathcal{H}$ such that

$$H\psi_n = \left[\alpha\left(\epsilon_n^+ + \epsilon_n^-\right) + \beta\right]\psi_n. \tag{12}$$